Energy Dissipation in an Oscillating Spherical Annulus Filled with a Viscous Fluid

H. F. KORMAN* AND L. R. KOVAL†
TRW Systems Group, Redondo Beach, Calif.

Introduction

Numerous occasions arise for which it is desirable to know the energy dissipated in an oscillating sphere filled with a viscous fluid. As discussed in Ref. 1, examples are the use of viscous dampers for damping the oscillations of a long flexible beam and for restricting the coupled two-body motion of two satellites connected by a long structure. A promising modification of the fluid-filled sphere concept arose out of the study of viscous dampers for spacecraft presented in Ref. 2. This concept consisted of placing a smaller sphere inside the main sphere, resulting in a spherical annulus filled with a viscous fluid. To analyze this configuration, it is necessary to calculate the energy dissipated in the fluid when the outer sphere is harmonically oscillating and the inner sphere is motionless. The following is a description of this calculation and a discussion of the results.

Formulation of the Problem

To compute the dissipation in the spherical annulus, the fluid motion induced in a viscous fluid lying between two concentric spheres of inner radius b and outer radius a is examined. The annulus is subjected to harmonic oscillations of the outer sphere about one of its diameters with the inner sphere motionless. For these conditions it is necessary to consider secondary flow effects and possible flow instabilities.[3] An analytical treatment might well require the type of expansions referred to in Ref. 4. Nevertheless, the good agreement between theory and experiment presented in Ref. 1 suggests that the same analytical approach for the oscillating annulus is worthy of pursuit. In view of the Ref. 4 comments the outer sphere oscillation amplitude Ω_0 will be restricted to be much less than the oscillation frequency p.

Thus the approximation shall be made that the only fluid velocity present is in the direction of increasing azimutha angle ϕ. The spherical geometry is shown in Fig. 1. Accordingly, $v_r = v_\theta = \partial P/\partial\phi = 0$ and $v_\phi = \psi(r,\theta,t)$ (whre v_i is the ith velocity component and P is the pressure). The Navier-Stokes equation of ψ is[5]

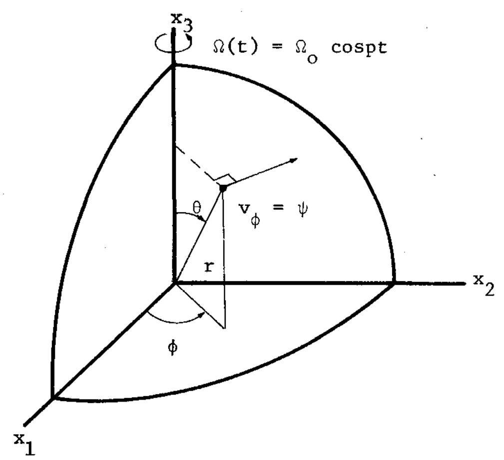

Fig. 1 Coordinate geometry.

$$\frac{\partial\psi}{\partial t} = \nu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - \frac{\psi}{r^2\sin^2\theta}\right] \tag{1}$$

where ψ is the fluid velocity component in the direction of increasing ϕ and ν is the kinematic viscosity. The appropriate boundary conditions are

$$\psi(a,\theta,t) = a\Omega_0\sin\theta\cos pt; \quad \psi(b,\theta,t) = 0 \tag{2}$$

Solution of the Boundary Value Problem

The solution of Eq. (1) and (2) can be obtained using Laplace transformation, as was done in Ref. 1. Writing $\bar{\psi}(r,\theta,s)$ for the transform of $\psi(r,\theta,t)$, s being the transformation variable, it is shown in Ref. 1 that $\bar{\psi}$ is given by

$$\bar{\psi}(r,\theta,s) = \sum_{l=1}^{\infty}[A_l j_l(w_r) + B_l n_l(w_r)]P_l^1(\cos\theta) \tag{3}$$

where $j_l(z)$ are the spherical Bessel functions of the first kind,[6] $n_l(z)$ are the spherical Bessel functions of the second kind, $P_l^1(\cos\theta)$ the associated Legendre polynomials,[7] and $w_r = (-s/\nu)^{1/2}r$. The constants A_l and B_l are determined from the transform of the boundary conditions.

Since $P_1^1(\cos\theta) = -\sin\theta$, and the $P_l^1(\cos\theta)$ are orthogonal functions, it is easily shown that

$$A_l = B_l = 0 \quad \text{for} \quad l > 1 \tag{4}$$

and

$$A_1 = a\Omega_0 s(s^2 + p^2)^{-1}n_1(w_b)/q(a,b) \tag{5}$$

$$B_1 = a\Omega_0 s(s^2 + p^2)^{-1}j_1(w_b)/q(a,b) \tag{6}$$

where

$$q(r,b) = j_1(w_r)n_1(w_b) - n_1(w_r)j_1(w_b) \tag{7}$$

$$w_a = (-s/\nu)^{1/2}a, \quad w_b = (-s/\nu)^{1/2}b \tag{8}$$

Hence,

$$\bar{\psi}(r,\theta,s) = -a\Omega_0 s(s^2 + p^2)^{-1}q(r,b)/q(a,b)\sin\theta \tag{9}$$

The inverse transform is given by

$$\psi(r,\theta,t) = \frac{a\Omega_0\sin\theta}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}(s^2+p^2)^{-1}q(r,b)/q(a,b)ds \tag{10}$$

It can be shown that the integrand for Eq. (10) contains no branch points. Hence the integration can be performed using Cauchy's integral theorem, and the method of residues. Since the zeros of $q(a,b)$ only give rise to the transient response of the damper and since only the asymptotic motions are of interest, the poles at $s = \pm ip$ need only be considered. Proceedings in the standard fashion,

$$\psi(r,\theta,t) = a\Omega_0\sin\theta\, Re[e^{ipt}q(Kr,Kb)/q(Ka,Kb)] \tag{11}$$

where

$$K = i(ip/\nu)^{1/2} \tag{12}$$

Dissipation of Energy

In spherical coordinates for the annular region, the rate of energy dissipation $\dot{E}$ is given by[5]

$$\dot{E} = 2\pi\mu\int_b^a\int_0^\pi\left[\left(\frac{\partial\psi}{\partial r} - \frac{\psi}{r}\right)^2 + \left(\frac{1}{r}\frac{\partial\psi}{\partial\theta} - \cos\theta\frac{\psi}{r}\right)^2\right]r^2\sin\theta d\theta dr \tag{13}$$

where μ is the dynamic viscosity. The second term in the integrand vanishes because ψ varies with $\sin\theta$. The energy

Received July 30, 1970; revision received March 12, 1971.
* Member of the Technical Staff, Advanced Technology Department. Member AIAA.
† Staff Engineer, Advanced Technology Department; now a Member of the Technical Staff, Advanced Analysis Department, Litton Ship Systems, Culver City, Calif. Member AIAA.

dissipated over one cycle, ΔE, is:

$$\Delta E = \int_0^{2\pi/p} \dot{E}dt = 2\pi\mu \int_0^{2\pi/p}\int_b^a\int_0^\pi \left(\frac{\partial\psi}{\partial r} - \frac{\psi}{r}\right)^2 \times r^2 \sin\theta d\theta dr dt \quad (14)$$

Upon substituting Eq. (11) into Eq. (14), it can be shown[8] (after a straight-forward but laborious integration) that Eq. (14) becomes

$$\Delta E = 64\pi^2\Omega_0^2 a^2 J/p(2p/\nu)^{1/2}x^4|D|^2 \quad (15)$$

where

$$J = \left(x\frac{1-3l}{12} - \frac{1}{z^3}\right)C^-(z-x) + \left(\frac{x^2}{24} + \frac{1-r}{z^2}\right)S^+(z-x) - \frac{(1-l)^2}{2z} \times C^+(z-x) + \frac{1-6l+6l^2}{12}S^-(z-x) \quad (16)$$

$$C^\pm(z-x) = \cosh(z-x) \pm \cos(z-x) \quad (17)$$

$$S^\pm(z-x) = \sinh(z-x) \pm \sin(z-x) \quad (18)$$

$$|D|^2 = \frac{4}{x^4z^4}\left\{\left[\frac{x^2z^2}{2} + (z-x)^2 + 2\right]\cosh(z-x) + (z-x)(xz-2)\sinh(z-x) - \left[\frac{x^2z^2}{2} - (z-x)^2 + 2\right]\cos(z-x) - (z-x)(xz+2)\sin(z-x)\right\} \quad (19)$$

with $z = a(2p/\nu)^{1/2} = x/l$, and $l = b/a$.

A nondimensional normalized energy dissipation per cycle, $E(\gamma,l)$, can be formed by dividing ΔE by $\frac{4}{3}\pi\rho_f a^3$ and by $\Omega_0^2a^2$, where ρ_f is the damping fluid density. If the Reynold's number parameter, $\gamma = pa^2/\nu = z^2/2$, and a reference mass (which is the mass of the fluid which would fill the entire sphere with the inner sphere removed), $M_h = \frac{4}{3}\pi\rho_f a^3$, are introduced

$$E(\gamma,l) = \Delta E/M_h\Omega_0^2a^2 = 48\pi J/(2)^{1/2}\gamma^{3/2}x^4|D|^2 \quad (20)$$

Discussion

The variation of E with γ and l is shown in Fig. 2. It can be seen that large values of E result for small values of γ and values of l close to unity. Large values of γ result in E approaching zero, while small values of l result in a limiting curve. This limiting curve corresponds to Fig. 2 of Ref. 1, which is for the outer sphere completely filled with fluid. These various limiting cases can be investigated individually.

Equation (20) can be simplified in the limit as $l \rightarrow 1$ for fixed γ. The resulting form is

$$\lim_{l\rightarrow 1} E = \frac{2\pi}{\gamma(1-l)} \quad (21)$$

It can be shown[8] that this is equivalent to a velocity profile which is linear in r. E is large for a value of γ of order one and l very close to unity. The curves in Fig. 2 for $l = 0.9999$, 0.999 and 0.99 obey Eq. (21), along with portions of the other curves. Hence for the small gap case, i.e., $l \rightarrow 1$, Eq. (21) is quite adequate for calculating E.

The large value of E for small γ can also be explained physically. Only the viscosity μ and the frequency p affect E without changing its normalizing factor. Large μ results in a small γ. A fluid which is very viscous requires a large energy input to move it. Hence a large viscosity results in a large energy dissipation. Small p results in a small γ. A low frequency corresponds to a long oscillation period. Hence a very small frequency requires the outer sphere to be moving for a long time and large energy input for one cycle.

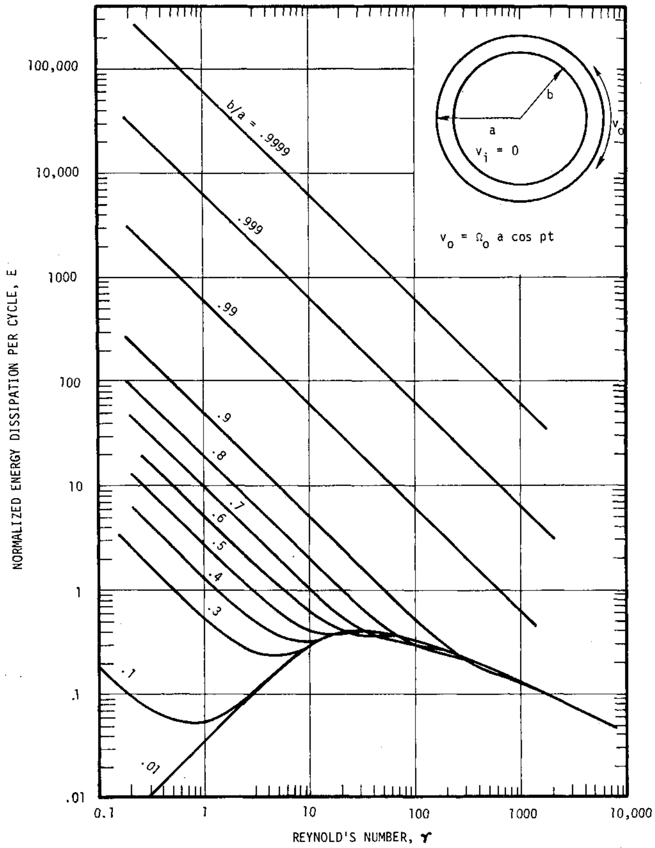

Fig. 2 The normalized energy dissipation in one cycle.

The result that E becomes small as γ becomes large is to be expected on physical grounds. A small viscosity μ results in very little fluid motion except near the outer sphere, and hence a small amount of energy is dissipated. A very large frequency p, has the outer sphere oscillating so rapidly that the bulk of the fluid only sees the average velocity of the outer sphere, which is zero. Since very little motion is imparted to the fluid, the energy dissipation is very small.

The equation for the limiting curve for small l can be determined from Eq. (20). The result is

$$\lim_{l\rightarrow 0} E = (2\pi/z^2DEN)[z^3S^-(z) - 6z^2C^+(z) + 12zS^+(z) - 12C^-(z)] \quad (22)$$

where

$$DEN = z^2C^+(z) - 2zS^+(z) + 2C^-(z) \quad (23)$$

Eq. (31) of Ref. 1 describes the limiting curve in terms of a doubly infinite series

$$[E]_{l=0} = \left(\frac{\pi^2z^4}{4DEN}\right)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{[(-1)^m + (-1)^n](-1)^{(m+n)/2}(z^2/8)^{m+n}}{(m-1)!(n-1)!(2m+2n+3)\Gamma(m+5/2)\Gamma(n+5/2)} \quad (24)$$

Equating the right sides of Eqs. (22) and (24) will yield the sum of the doubly infinite series in Eq. (24). This is a rather interesting byproduct of the present analysis.

A general comment should be made about the behavior of $E(\gamma,l)$. It is always determined by a tradeoff between Reynold's number effects and inner sphere effects. The assumption of an inertially fixed inner sphere in the analysis vitally affects the low γ and $l \rightarrow 1$ results of E becoming very large.

References

[1] Aprahamian, R., Johnson, R. L., and Koval, L. R., "Energy Dissipation in an Oscillating Sphere Filled with a Viscous Fluid," *AIAA Journal*, Vol. 7, No. 9, Sept. 1969, pp. 1793–1796.

[2] "Study of Fluid Dampers for Booms," Rept. 09871.00-6011-TO-00, Contract NAS 5-10496, July 1968, Advanced Technology Dept., TRW Systems Group, Redondo Beach, Calif.

[3] Zierep, J. and Sawatzki, O., "Three Dimensional Instabilities and Vortices Between Two Rotating Spheres," *Second International Conference on Numerical Methods in Fluid Dynamics*, Berkeley, Calif., Sept. 1970.

[4] Lee, D. A., "Comment on 'Energy Dissipation in an Oscillating Sphere Filled with a Viscous Fluid'," *AIAA Journal*, Vol. 8, No. 10, Oct. 1970, p. 1919.

[5] Landau, L. D. and Lifshitz, E. M., *Fluid Mechanics*, Addison-Wesley, Reading, Mass., 1959, pp. 52–54.

[6] Abramovitz, M. and Stegun, K. A., *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics, Ser. 55, 1965, pp. 437–438.

[7] Jackson, J. D., *Classical Electrodynamics*, Wiley, New York, 1962, pp. 56–66.

[8] "Study of a Spherical Annular Boom Damper," Rept. 12726-6001-RO-00, Contract NAS 5-11630, July 1969, Advanced Technology Department, TRW Systems Group, Redondo Beach, Calif.

Stiffness Analysis Using Multi global Axes System

Fred W. Beaufait*

Vanderbilt University, Nashville, Tenn.

Introduction

THE analysis of a structural system by the stiffness method can be achieved from the set of dependent equations

$$[S_c][\Delta_c] = [JL_c] + [R_c] \quad (1)$$

which describe the static equilibrium of the numerous joints within the structure: $[S_c]$ is the complete structure stiffness matrix, $[\Delta_c]$ is the complete joint displacement matrix, $[JL_c]$ is the complete joint load matrix and $[R_c]$ is the complete support reaction matrix. This matrix equation [Eq. (1)] yields the set of independent equations

$$[S_{uu}][\Delta_u] = [JL_u] \quad (2)$$

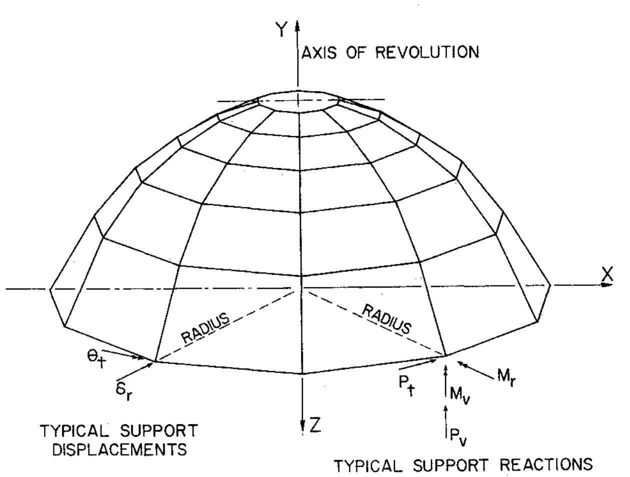

Fig. 1 Rigid framed dome.

Paper received July 1, 1970; revision received February 8, 1971.

* Associate Professor of Civil Engineering.

which can be solved to evaluate the unrestrained components of joint displacement of the structure for specified loading conditions, and

$$[R_r] = [S_{ru}][S_{uu}]^{-1}[JL_u] - [JL_r] \quad (3)$$

which can be used to evaluate the support reactions of the structure. In Eqs. (2) and (3), $[S_{uu}]$ is the matrix of structure stiffness coefficients corresponding to the unrestrained components of joint displacement of the actual structure resulting from unit values of each of these components of joint displacements applied to the restrained structure; $[S_{ru}]$ is the matrix of structure stiffness coefficients corresponding to restrained components of joint displacement of the actual structure resulting from unit values of unrestrained components of joint displacements of the actual structure applied to the restrained structure; $[\Delta_u]$ is the matrix of unrestrained independent components of joint displacement; $[JL_u]$ and $[JL_r]$ are the matrices of joint loads corresponding to the unrestrained and restrained components of joint displacement, respectively; and $[R_r]$ is the matrix of the components of support reactions. The derivation and application of Eqs. (1–3) for the analysis of a structural system can be found in Refs. 1–3 and will not be discussed herein.

It has been general practice when developing the various matrices of Eq. (1) to describe all of the components of joint displacement, joint load and support reaction with respect to a single set of coordinate axes, i.e., the global or reference axes. Consequently, in order to maintain the continuity of Eq. (1) the structure stiffness coefficients which make up the matrix $[S_c]$ must be evaluated with respect to this set of reference axes; this means that the element stiffness coefficients which relate the end actions and the end displacements of an individual element, must be defined with respect to this single set of reference axes. Unfortunately, there are instances where it is either impossible or impractical to develop the set of independent joint equilibrium equations of Eq. (2) with respect to a single set of reference axes. Consider, for example, the rigid framed dome structure of Fig. 1. If it were supported at each joint of the lower ring such that the joints were free to translate radially in the plane of the ring (δ_r) and rotate about an axis in the plane of the ring perpendicular to the radius of the ring at the joint (θ_t), it would be most difficult to describe the components of displacement of all of the boundary joints with respect to a single set of rectangular Cartesian coordinate axes

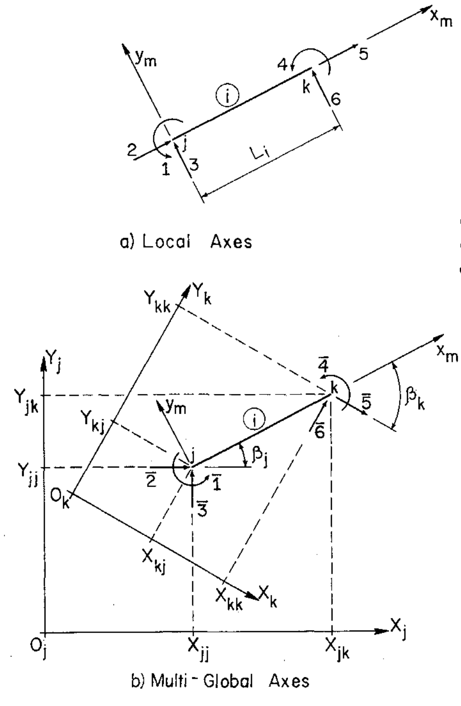

Fig. 2 Identification of axes and end actions or displacements for beam element.